PUTNAM PRACTICE SET 8

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Problem 1. Find all real numbers a with the property that the equation

$$x^2 - 2x \cdot [x] + x - a = 0$$

has two distinct nonnegative real roots.

Solution. We let x_1 be a solution of this equation and we let n_1 be the integer part of x_1; then $n_1 \geq 0$ by our assumption about the solutions of our equation. So, $x_1 = n_1 + r_1$, where $0 \leq r_1 < 1$. Then we have

$$a = (n_1 + r_1)^2 - 2n_1(n_1 + r_1) + n_1 + r_1 = (n_1 - n_1^2) + (r_1^2 + r_1).$$

So, we are looking for those numbers a for which there exist another solution $x_2 = n_2 + r_2$ (also with $0 \leq r_2 < 1$) such that again

$$a = (n_2 - n_2^2) + (r_2^2 + r_2).$$

So, the existence of two solutions yields that

$$\left|(n_2^2 - n_2) - (n_1^2 - n_1)\right| = \left|(r_2^2 + r_2) - (r_1^2 + r_1)\right| < 2.$$

Without loss of generality, we may assume $n_2 \geq n_1$. Now, if $n_2 = n_1$, then we would actually get that also $r_2^2 + r_2 = r_1^2 + r_1$ and therefore, either $r_2 = r_1$ (which leads to $x_2 = x_1$ and thus, a contradiction that the two solutions are distinct), or $r_2 + r_1 = -1$, which is obviously a contradiction. Therefore, from now on, we assume that $n_2 > n_1$. However, in this case, we have

$$(n_2^2 - n_2) - (n_1^2 - n_1) = (n_2 - n_1)(n_2 + n_1 - 1) \geq n_2 + n_1 - 1 \geq 2,$$

unless $n_1 = 0$ and $n_2 = 1$. Now, if $n_1 = 0$ and $n_2 = 1$, we get that $r_2^2 + r_2 = r_1^2 + r_1$ and hence $r_2 = r_1$. So, we have indeed two solutions in this case $x_1 = r$ and $x_2 = 1 + r$, where $r \in [0, 1)$. We see then that $a = r^2 + r$ and so, the values of a which allow the existence of two nonnegative solutions to the given equations are all the real numbers in the interval $[0, 2)$.

Problem 2. Let $k, n \in \mathbb{N}$ such that $n \geq k^3 + 1$. We partition $\{1, 2, 3, \ldots, 2n\}$ into k (disjoint) subsets, i.e.,

$$\{1, 2, 3, \ldots, 2n\} = M_1 \cup M_2 \cup \cdots \cup M_k.$$

Prove that there exist $i, j \in \{1, \ldots, k\}$ (possibly $i = j$) and there exist $(k + 1)$ distinct numbers

$$x_1, \ldots, x_{k+1} \in \{1, \ldots, n\}$$

such that $2x_1, \ldots, 2x_{k+1} \in M_i$ and $2x_1 - 1, \ldots, 2x_{k+1} - 1 \in M_j$.

Solution. For each $i = 1, \ldots, k$, we let

$$O_i := \{1 \leq j \leq n \colon 2j - 1 \in M_i\} \text{ and } E_i := \{1 \leq j \leq n \colon 2j \in M_i\}.$$

Then our hypothesis yields that

$$\cup_{i=1}^k O_i = \cup_{i=1}^k E_i = \{1, \dots, n\}.$$

Also, $O_i \cap O_j = E_i \cap E_j = \emptyset$ for each $i \neq j$. The conclusion in our problems asks for the existence of some $i, j \in \{1, \dots, k\}$ such that $\#(E_i \cap O_i) \geq k$. However (using the pigeonhole principle), our hypothesis yields the existence of some $i \in \{1, \dots, k\}$ such that $\#E_i \geq k^2 + 1$ (since there are $n \geq k^3 + 1$ in the union of all sets E_i and there are only k such sets). Then again using the pigeonhole principle, we obtain that there must be some $j \in \{1, \dots, k\}$ such that $E_i \cap O_j$ has at least $k+1$ elements since E_i has in common with all the sets O_j at least $k^2 + 1$ elements and there are only k such sets O_j.

Problem 3. In a finite sequence $\{x_n\}_{1 \leq n \leq m}$ of integers, the sum of each consecutive 5 numbers in the sequence is negative, while the sum of each 7 consecutive numbers in the sequence is positive. Find with proof the largest value for m.

Solution. We claim that $m = 10$. First of all, if there were a sequence of 11 numbers with this property, then we let $y_n := \sum_{i=1}^n x_i$ for each $n = 0, \dots, m$ (with the convention that $y_0 = 0$). Then our hypotheses yield

$$y_0 < y_7 < y_2 < y_9 < y_4 < y_{11} < y_6 < y_1 < y_8 < y_3 < y_{10} < y_5 < y_0,$$

which is a contradiction. Now, in order to construct a sequence with 10 terms, then with the same notation as above, we see that

$$0 = y_0 < y_7 < y_2 < y_9 < y_4$$

and

$$y_6 < y_1 < y_8 < y_3 < y_{10} < y_5 < y_0 = 0.$$

So, in order to construct the sequences $\{x_n\}_{n=1}^{10}$, all we need is a sequence of numbers $\{y_n\}_{n=0}^{10}$ satisfying the above inequalities since then $x_n = y_n - y_{n-1}$ for each $n = 1, \dots, 10$.

Problem 4. Let $n \in \mathbb{N}$ and let S be the set of all tuples $(a_1, \dots, a_n)$ satisfying $a_i \in \{-1, 1\}$ for each $i = 1, \dots, n$. For two elements $x, y \in S$ of the form $x := (a_1, \dots, a_n)$ and $y := (b_1, \dots, b_n)$, we define

$$x \cdot y := (a_1 b_1, a_2 b_2, \cdots, a_n b_n) \in S.$$

Let $B \subseteq S$ be a subset with $k \geq 1$ elements. Prove that there exists some $x_0 \in S$ such that the subset of S given by

$$x_0 \cdot B := \{x_0 \cdot y \colon y \in B\}$$

intersects B in a set with at most $\frac{k^2}{2^n}$ elements.

Solution. We observe that for each pair of elements $(y, z) \in B \times B$ there exists a unique $x_0 \in S$ such that $x_0 y = z$. Therefore, since there are k^2 such pairs, we have

$$\sum_{x \in S} \#\left((x \cdot B) \cap B\right) \leq k^2$$

and since there exist 2^n elements in S, the desired conclusion follows.